

# ON ENTROPY BUMPS FOR CALDERÓN-ZYGMUND OPERATORS

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**ABSTRACT.** We study two weight inequalities in the recent innovative language of ‘entropy’ due to Treil-Volberg. The inequalities are extended to  $L^p$ , for  $1 < p \neq 2 < \infty$ , with new short proofs. A result proved is as follows. Let  $\varepsilon$  be a monotonic increasing function on  $(1, \infty)$  which satisfy  $\int_1^\infty \frac{dt}{\varepsilon(t)t} = 1$ . Let  $\sigma$  and  $w$  be two weights on  $\mathbb{R}^d$ . If this supremum is finite, for a choice of  $1 < p < \infty$ ,

$$\sup_{Q \text{ a cube}} \left[ \frac{\sigma(Q)}{|Q|} \right]^{p-1} \frac{\int_Q M(\sigma \mathbf{1}_Q)}{\sigma(Q)} \cdot \frac{w(Q)}{|Q|} \left[ \frac{\int_Q M(w \mathbf{1}_Q)}{w(Q)} \right]^{p-1} < \infty,$$

then any Calderón-Zygmund operator  $T$  satisfies the bound  $\|T_\sigma f\|_{L^p(w)} \lesssim \|f\|_{L^p(\sigma)}$ .

## 1. INTRODUCTION

We are concerned with two weight inequalities, and this general question: What is the ‘simplest’ condition which is analogous to the Muckenhoupt  $A_p$  condition, and is sufficient for a two weight inequality to hold for all Calderón-Zygmund operators? This question arose shortly after the initial successes of the Muckenhoupt [19], and Hunt-Muckenhoupt-Wheeden [4]. And, much work was following the lines of [21], which lead to the notion of testing the density of the weights in function spaces of slightly stronger norms. This theme has been investigated by many authors, with motivations coming from potential applications in different settings where Calderón-Zygmund operators appear, see for instance [3, 26] for two disparate applications. More relevant citations are in the introduction to [2], for instance.

Concerning the maximal operator itself, the finest result in this direction is due to Pérez [22]: A sharp integrability condition is used to describe a class of Orlicz spaces, and an  $A_p$  like condition, which is a sufficient condition for a two weight inequality for the maximal function. We do not recall the exact conditions, since the entropy conditions used in this paper allow a shorter presentation of more general results. For the maximal function, this is Theorem 1.2 below.

Pérez also raised two conjectures concerning singular integrals, on being the so-called two-bump conjecture resolved in [15, 20], and the so-called separated bump conjecture which is unresolved, [2, 6].

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Several recent papers have focused on the role of the  $A_\infty$  constant in completing these estimates. This theme was started in [13], and was further quantified in several papers [8–12, 17, 18].

Recently, Treil-Volberg [25] combined these two trends in a single approach, which they termed the *entropy bounds*, and as is explained in [25, §2], this approach yields (slightly) stronger results than that of the Orlicz function approach. In this paper, we will extend their results to the  $L^p$ -setting, using very short proofs. The main results are as follows. Throughout, let

$$\rho_\sigma(Q) = \frac{\int_Q M(\sigma \mathbf{1}_Q) \, dx}{\sigma(Q)}, \quad \rho_{\sigma, \varepsilon}(Q) = \rho_\sigma(Q) \varepsilon(\rho_\sigma(Q)),$$

where  $\varepsilon$  will be an increasing function on  $[1, \infty)$ . But, if the role of the weight  $\sigma$  is understood, it is suppressed in the notation. Define

$$(1.1) \quad [\sigma, w]_{p, \varepsilon} := \sup_{Q \text{ a cube}} \rho_{\sigma, \varepsilon}(Q) \langle \sigma \rangle_Q^{p-1} \langle w \rangle_Q.$$

Throughout,  $\langle f \rangle_Q = |Q|^{-1} \int_Q f(x) \, dx$ . In this Theorem, we extend the result of Pérez [22] for the maximal function to the entropy language.

**Theorem 1.2.** *Let  $\sigma$  and  $w$  be two weights with densities, and  $1 < p < \infty$ . Let  $\varepsilon$  be a monotonic increasing function on  $(1, \infty)$  which satisfies  $\int_1^\infty \frac{dt}{\varepsilon(t)t} = 1$ . There holds*

$$(1.3) \quad \|M_\sigma : L^p(\sigma) \mapsto L^p(w)\| \lesssim [\sigma, w]_{p, \varepsilon}^{1/p}.$$

Here, and throughout, we use the notation  $M_\sigma f = M(\sigma f)$ , so that inequalities are stated in a self-dual way.

Concerning Calderón-Zygmund operators, the case of  $p = 2$  below is [25, Thm. 2.5]. It is slightly stronger than the two bump conjecture proved in [15, 20].

**Theorem 1.4.** *Let  $\sigma$  and  $w$  be two weights with densities, and  $1 < p < \infty$ . Let  $\varepsilon$  be a monotonic increasing function on  $(1, \infty)$  which satisfies  $\int_1^\infty \frac{dt}{\varepsilon(t)t} = 1$ . Define*

$$(1.5) \quad [\sigma, w]_p := \sup_{Q \text{ a cube}} \langle \sigma \rangle_Q^{p-1} \rho_{\sigma, \varepsilon}(Q) \langle w \rangle_Q \rho_{w, \varepsilon}(Q)^{p-1}$$

For any Calderón-Zygmund operator, there holds

$$\|T_\sigma : L^p(\sigma) \rightarrow L^p(w)\| \lesssim C_T [\sigma, w]_p^{1/p}.$$

The constant  $C_T$  is defined in (2.1).

In the condition (1.5) above, both of the weights  $\sigma$  and  $w$  are ‘bumped.’ Below, the bump is applied to each weight separately, hence the name *separated bump* condition. The case  $p = 2$  below corresponds to [25, Thm 2.6]. It is slightly stronger than the corresponding results proved in [6].

**Theorem 1.6.** *Let  $\sigma$  and  $w$  be two weights with densities, and  $1 < p < \infty$ . Let  $\varepsilon_p, \varepsilon_{p'}$  be two monotonic increasing functions on  $(1, \infty)$  which satisfy  $\int_1^\infty \varepsilon_p(t)^{-1/p} \frac{dt}{t} = 1$ , and similarly for  $\varepsilon_{p'}$  with root  $1/p'$ . For any Calderón-Zygmund operator, there holds*

$$\|T_\sigma : L^p(\sigma) \rightarrow L^p(w)\| \lesssim C_T \{ [\sigma, w]_{p, \varepsilon_p}^{1/p} + [w, \sigma]_{p', \varepsilon_{p'}}^{1/p'} \}.$$

The terms involving the weights is defined in (1.1), and the constant  $C_T$  is defined in (2.1).

One should not fail to note that the integrability condition imposed on  $\varepsilon_p(t)^{-1}$  is stronger than in Theorem 1.4. It is not known if the condition in Theorem 1.6 is the sharp. Furthermore, one can see that the two Theorems are not strictly comparable: There are examples of weights that meet the criteria of one Theorem, but not the other.

The method of proof we use is, like Lerner [16], to reduce to sparse operators. With the recent argument of one of us, [7], this reduction now applies more broadly, namely it applies to (a) Calderón-Zygmund operators on Euclidean spaces as stated above; (b) non-homogeneous Calderón-Zygmund operators; and (c) general martingales. See [7] for some details.

After the reduction to sparse operators, we use arguments involving pigeon-holes, stopping times, reduction to testing conditions, and an  $\Lambda_p$ - $\Lambda_\infty$  inequality. These are the shortest proofs we could find.

## 2. NOTATION, BACKGROUND

Constants are suppressed: By  $A \lesssim B$ , it is meant that there is an absolute constant  $c$  so that  $A \leq cB$ . We will use the notation  $A \sim B$  to mean that  $A \leq B \leq 2A$ .

We say that  $K : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  is a *Calderón-Zygmund kernel* if for some constants and  $C_K > 0$ , and  $0 < \eta < 1$ , such that these conditions hold: For  $x, x', y \in \mathbb{R}^d$

$$\begin{aligned} \|K(\cdot, \cdot)\|_\infty &< \infty, \\ |K(x, y)| &< C_K |x - y|^{-d}, \quad x \neq y, \\ |K(x, y) - K(x', y)| &< C_K \frac{|x - x'|^\eta}{|x - y|^{d+\eta}}, \quad \text{if } 2|x - x'| < |x - y|, \end{aligned}$$

and a fourth condition, with the roles of the first and second coordinates of  $K(x, y)$  reversed also holds. These are typical conditions, although in the first condition, we have effectively truncated the kernel, at the diagonal and infinity. The effect of this is that we needn't be concerned with principal values.

Given a Calderón-Zygmund kernel  $K$  as above, we can define

$$Tf(x) := \int K(x, y)f(y) \, dy$$

which is defined for all  $f \in L^2$  and  $x \in \mathbb{R}^d$ . We say that  $T$  is a *Calderón-Zygmund operator*, since it necessarily extends to a bounded operator on  $L^2(\mathbb{R}^d)$ . We define

$$(2.1) \quad C_T := C_K + \|T : L^2 \rightarrow L^2\|.$$

It is well-known that  $T$  is also bounded on  $L^p$ ,  $1 < p < \infty$ , with norm controlled by  $C_T$ .

We use the recent inequality [7], which gives *pointwise control* of a Calderón-Zygmund operator by a sparse operator.  $S$  is a *sparse operator* if  $Sf = \sum_{Q \in \mathcal{Q}} \langle f \rangle_Q 1_Q$ , where  $\mathcal{S}$  is a collection of

dyadic cubes for which

$$\left| \bigcup_{Q' \in \mathcal{S}: Q' \subsetneq Q} Q' \right| \leq \frac{1}{2}|Q|.$$

We will also refer to  $\mathcal{S}$  as *sparse*, and will typically suppress the dependence of  $\mathcal{S}$  on the sparse collection  $\mathcal{S}$ . Trivially, any subset of a sparse collection is sparse. By abuse of notation, if an operator is sparse with respect to a choice of grid, we call it sparse.

A sparse operator is bounded on all  $L^p$ , and in fact, is a ‘*positive* dyadic Calderón-Zygmund operator.’ And the class is sufficiently rich to capture the norm behavior of an arbitrary Calderón-Zygmund operator.

**Theorem A.** [7, Thm 5.2] *For all  $T$  and compactly supported  $f \in L^1$ , there are at most  $N \leq 3^d$  sparse operator  $S_1, \dots, S_N$  (associated to distinct choices of grids) so that  $|Tf| \lesssim \sum_{n=1}^N S|f|$ .*

As a consequence, we see that it suffices to prove our main Theorems for sparse operators.

### 3. PROOF OF THEOREM 1.2

We prove the maximal function estimate (1.3). It suffices to prove the theorem with the maximal function replaced by a dyadic version, since it is a classical fact that in dimension  $d$ , there are at most  $3^d$  choices of shifted dyadic grids  $\mathcal{D}_j$ , for  $1 \leq j \leq 3^d$ , which approximate any cube in  $\mathbb{R}^d$ .

By Sawyer’s characterization [23] of the two weight maximal function inequality, it suffices to check that inequality for  $f = \mathbf{1}_{Q_0}$ , and any dyadic cube  $Q_0$ . Namely, we should prove

$$\int_{Q_0} M(\sigma \mathbf{1}_{Q_0})^p dw \lesssim [\sigma, w]_{p, \varepsilon}(\sigma(Q_0)).$$

To do so, let  $\mathcal{S}$  be a sequence of stopping cubes for  $\sigma$ , defined as follows. The root of  $\mathcal{S}$  is  $Q_0$ , and if  $S \in \mathcal{S}$ , the maximal dyadic cubes  $Q \subset S$  such that  $\langle \sigma \rangle_Q > 4\langle \sigma \rangle_S$  are also in  $\mathcal{S}$ . Note that this is a sparse collection of cubes. Then, we have

$$\mathbf{1}_{Q_0} \cdot M(\sigma \mathbf{1}_{Q_0}) \lesssim \sum_{S \in \mathcal{S}} \langle \sigma \rangle_S \mathbf{1}_{E_S}$$

where  $E_S := S \setminus \bigcup \{S' \in \mathcal{S} : S' \subsetneq S\}$ . The collection  $\mathcal{S}$  is sparse, and the sets  $E_S$  are pairwise disjoint, hence,

$$\int_{Q_0} M(\sigma \mathbf{1}_{Q_0})^p dw \lesssim \sum_{S \in \mathcal{S}} \langle \sigma \rangle_S^p w(S).$$

The sparse collection  $\mathcal{S}$  is divided into collations  $\mathcal{S}_{\alpha, r}$ , for  $\alpha \in \mathbb{Z}$  and  $r \in \mathbb{N}$  defined by  $S \in \mathcal{S}_{\alpha, r}$  if and only if

$$2^\alpha \sim \langle \sigma \rangle_S^{p-1} \langle w \rangle_Q \rho_{\sigma, \varepsilon}(Q), \quad \text{and} \quad 2^r \sim \rho(Q).$$

Notice that  $\mathcal{S}_{\alpha, r}$  is empty if  $[\sigma, w]_{p, \varepsilon} < 2^{\alpha-1}$ .

Then, estimate as below, holding  $\alpha$  and  $r$  constant.

$$\begin{aligned} \sum_{S \in \mathcal{S}_{\alpha,r}} \langle \sigma \rangle_S^p w(S) &\lesssim 2^\alpha \sum_{S \in \mathcal{S}_{\alpha,r}} \frac{\sigma(S)}{2^r \varepsilon(2^r)} \\ &\lesssim 2^\alpha \sum_{S \text{ is maximal in } \mathcal{S}_{\alpha,r}} \frac{\int_S M(\sigma \mathbf{1}_S)}{2^r \varepsilon(2^r)} \lesssim 2^\alpha \frac{\sigma(Q_0)}{\varepsilon(2^r)}. \end{aligned}$$

Notice that sparsity is essential to the domination of the sum by the maximal function in the second line. To sum this over  $r \in \mathbb{N}$ , we need the integrability condition  $\int_1^\infty \frac{dt}{\varepsilon(t)t} = 1$ . Then, take  $p$ th roots, and sum over appropriate  $\alpha \in \mathbb{Z}$  to conclude the proof.

#### 4. PROOF OF THEOREM 1.4

Fix a sparse collection  $\mathcal{S}$  so that for all cubes  $Q \in \mathcal{S}$  there holds, for some  $\alpha \in \mathbb{Z}$ ,

$$2^\alpha \sim \langle \sigma \rangle_Q^{p-1} \rho_{\sigma,\varepsilon}(Q) \langle w \rangle_Q \rho_{\sigma,\varepsilon,p'}(Q)^{p-1}$$

Here,  $2^{\alpha-1} \leq \lfloor \sigma, w \rfloor_p$ . In this case, we will verify that the norm of the associated sparse operator is bounded as by  $\lesssim 2^{\alpha/p}$ . This estimate is clearly suitable in relevant  $\alpha \in \mathbb{Z}$ .

The proof is by duality. Thus, for  $f \in L^p(\sigma)$  and  $g \in L^{p'}(w)$ , we bound the pairing  $\langle S(\sigma f), gw \rangle$ . In so doing, we will write

$$\langle f\sigma \rangle_Q = \langle f \rangle_Q^\sigma \langle \sigma \rangle_Q,$$

where  $\langle f \rangle_Q^\sigma$  is the average of  $f$  relative to weight  $\sigma$  on the cube  $Q$ . Then,

$$\begin{aligned} 2^{-\alpha/p} \langle S(\sigma f), gw \rangle &= 2^{-\alpha/p} \sum_{Q \in \mathcal{S}} \langle \sigma f \rangle_Q \langle gw \rangle_Q \cdot |Q| \\ &= \sum_{Q \in \mathcal{S}} \langle f \rangle_Q^\sigma \langle \sigma \rangle_Q^{1/p} \left\{ \frac{\langle \sigma \rangle_Q^{1/p'} \langle w \rangle_Q^{1/p}}{2^{\alpha/p}} \right\} \langle w \rangle_Q^{1/p'} \langle g \rangle_Q^w \cdot |Q| \\ &\lesssim \sum_{Q \in \mathcal{S}} \langle f \rangle_Q^\sigma \frac{\sigma(Q)^{1/p}}{\rho_{\sigma,\varepsilon}(Q)^{1/p}} \cdot \langle g \rangle_Q^w \frac{w(Q)^{1/p'}}{\rho_{w,\varepsilon}(Q)^{1/p'}}. \end{aligned}$$

Apply Hölder's inequality to the last expression. It clearly suffices to show that

$$\sum_{Q \in \mathcal{S}} (\langle f \rangle_Q^\sigma)^p \frac{\sigma(Q)}{\rho_{\sigma,\varepsilon}(Q)} \lesssim \|f\|_{L^p(\sigma)}^p,$$

and similarly for  $g$ .

This last expression is a Carleson embedding inequality. It is well known that it suffices to check this inequality for  $f = \mathbf{1}_{Q_0}$ , for  $Q_0 \in \mathcal{S}$ , and then one can impose the assumption that  $Q_0$  is the maximal element in  $\mathcal{S}$ . But notice that the sum to control is then

$$\sum_{Q \in \mathcal{S}} \frac{\sigma(Q)}{\rho_{\sigma,\varepsilon}(Q)} \lesssim \sum_{r=1}^{\infty} \sum_{\substack{Q \in \mathcal{S} \\ \rho_{\sigma,\varepsilon}(Q) \sim 2^r}} \frac{\sigma(Q)}{2^r \varepsilon(2^r)}$$

$$\lesssim \sum_{r=1}^{\infty} \sum_{\substack{Q \text{ maximal s.t.} \\ Q \in \mathcal{S}, \rho_{\sigma}(Q) \sim 2^r}} \frac{\int_Q M(\sigma \mathbf{1}_Q) dx}{2^r \varepsilon(2^r)} \lesssim \sigma(Q_0) \sum_{r=0}^{\infty} \frac{1}{\varepsilon(2^r)}.$$

The middle inequality follows from sparseness. The last sum over  $r$  should be finite, which is our integrability condition on  $\varepsilon$ :  $\int_1^{\infty} \frac{dt}{t\varepsilon(t)} = 1$ . The proof is complete.

## 5. PROOF OF THEOREM 1.6

The key fact is this Lemma. In the current setting, it originates in [12], though we give a more convenient reference below. Notice that the bound on the right in the estimates below are specific to the sparse collection being used.

**Lemma 5.1.** [9, Prop. 5.3] *Let  $\mathcal{S}$  be a sparse collection of cubes all contained in a cube  $Q_0$ , defining a sparse operator  $S$ . For two weights  $\sigma, w$ , there holds*

$$(5.2) \quad \int_{Q_0} (S\sigma \mathbf{1}_{Q_0})^p dw \lesssim A_p(\mathcal{S}) A_{\infty}(\mathcal{S}) \sigma(Q_0),$$

where

$$A_p(\mathcal{S}) := \sup_{Q \in \mathcal{S}} \langle \sigma \rangle_Q^{p-1} \langle w \rangle_Q, \quad A_{\infty}(\mathcal{S}) := \sup_{Q \in \mathcal{S}} \frac{\int_Q M(\mathbf{1}_Q \sigma) dx}{\sigma(Q)}.$$

We need this consequence of the two weight theory of Sawyer [24]. Namely, since a sparse operator is positive, it suffices to verify a testing condition: For any dyadic cube  $Q_0$ ,

$$\int_{Q_0} \left| \sum_{Q \in \mathcal{S}: Q \subset Q_0} \langle \sigma \rangle_Q \mathbf{1}_Q \right|^p dw \lesssim [\sigma, w]_{p, \varepsilon_p} \sigma(Q_0).$$

The dual inequality will also hold, and so complete the proof of Theorem 1.6. The dyadic version of Sawyer's Theorem is the main result in [14], and an efficient proof is given on the last page of Hytönen's survey [5].

For integers  $a \in \mathbb{Z}$ , and  $r \in \mathbb{N}$  set  $\mathcal{S}_{a,r}$  to be all those cubes  $Q \in \mathcal{S}$  such that  $Q \subset Q_0$ ,

$$2^a \sim \rho_{\sigma, \varepsilon_p}(Q) \langle \sigma \rangle_Q^{p-1} \langle w \rangle_Q, \quad \text{and} \quad 2^r \sim \frac{\int_Q M(\mathbf{1}_Q \sigma) dx}{\sigma(Q)}.$$

Of course this collection is empty if  $[\sigma, w]_{p, \varepsilon_p} < 2^{a+1}$ . By construction,  $A_{\infty}(\mathcal{S}_{a,r}) \lesssim 2^r$ , and

$$A_p(\mathcal{S}_{a,r}) \lesssim \frac{2^a}{\rho_{\sigma, \varepsilon_p}(Q)} \simeq \frac{2^a}{2^r \varepsilon_p(2^r)}.$$

Thus, from (5.2), we have

$$\int_{Q_0} \left[ \sum_{Q \in \mathcal{S}_{a,r}} \langle \sigma \rangle_Q \mathbf{1}_Q \right]^p dw \lesssim A_p(\mathcal{S}_{a,r}) A_{\infty}(\mathcal{S}_{a,r}) \sigma(Q_0) \lesssim \frac{2^a}{\varepsilon_p(2^r)} \sigma(Q_0).$$

Take  $p$ th root, and sum over the relevant  $a \in \mathbb{Z}$ , and  $r \in \mathbb{N}$ . The sum over  $r$  is finite since  $\int_1^{\infty} \frac{dt}{t\varepsilon_p(t)^{1/p}} = 1$ , completing the proof.

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